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# On simple Lie algebras over a field of characteristic 2

Alexander Grishkov [1]

*University of Sao Paulo, Brazil*

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ABSTRACT

We prove that the simple Lie algebras constructed by G. Jurman (2004) in [2] are isomorphic to Hamiltonian algebras. As a corollary we answer all questions formulated in G. Jurman (2004) [2] about isomorphisms of these algebras.



## 1. Jurman's simple Lie algebras

In [2] G. Jurman constructed a family of simple Lie algebras $B(a,b)$ over a field $F$ of characteristic 2, $a,b \in \mathbf{Z}$, $a > 1$, $b > 0$. He formulated the following two questions:

1. Are the algebras $B(a+1,b)$ and $B(b+1,a)$ isomorphic?
2. Are the algebras $B(2,1)$ and $K_{14}$ isomorphic? Here $K_{14}$ is the Kaplansky 14-dimensional simple Lie algebra [3].

We prove that $B(a,b)$ is isomorphic to the Hamiltonian algebra of Cartan type $H(2,(a,b+1),\omega_0)^{(2)}$, where $\omega_0 = dx_1 \wedge dx_2$ is the standard volume form. The algebra $H(2,(a,b+1),\omega_0)^{(2)}$, may be realized as the space of truncated polynomials with the basis

$$\left\{x_1^i x_2^j \mid 0 \leqslant i < 2^a,\ 0 \leqslant j < 2^{b+1},\ (i,j) \neq (0,0), \left(2^a-1, 2^{b+1}-1\right)\right\}$$

and multiplication

$$[f,g] = \partial_1 f \partial_2 g + \partial_1 g \partial_2 f.$$

*E-mail address:* shuragri@gmail.com.

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In particular, we get

$$\left[x_1^{(i)}x_2^{(j)}, x_1^{(p)}x_2^{(q)}\right] = \left(\binom{i+p-1}{i}\binom{j+q-1}{j-1} + \binom{i+p-1}{i-1}\binom{j+q-1}{j}\right)x_1^{(i+p-1)}x_2^{(j+q-1)}.$$

In [2] the following basis of $B(a,b)$ was constructed $\{y_i, z_i \mid -1 \leqslant i \leqslant 2^{a+b}-3\}$, where

$$[y_i, y_j] = a_{ij}y_{i+j}, \qquad [y_i, z_j] = a_{ij}z_{i+j}, \qquad [z_i, z_j] = b_{ij}y_{i+j-2^{a-1}+1}, \tag{1}$$

$$a_{ij} = \binom{i+j+2}{i+1}, \qquad b_{ij} = \binom{i+j+2-2^a}{i+1} + \binom{i+j+3-2^a}{j+1}. \tag{2}$$

Denote $z = z_{2^{a-1}-1}$ and $Y_i = y_i + [y_i, z]$, $Z_i = z_i + [z_i, z]$. We suppose that $Y_i = Z_i = 0$, if $i > 2^{a+b}-3$ or $i < -1$.

**Theorem 1.** *The algebras $B(a,b)$ and $H(2,(a,b+1),\omega_0)^{(2)}$, are isomorphic.*

**Proof.** Let $B(a,b) = B(a,b)_0 \oplus B(a,b)_1$ be Fitting decomposition with respect to $ady_0$. Then by (1)–(2) we get

$$B(a,b)_i = \{x \mid [x, y_0] = ix\} = Span_k\{y_j, z_j \mid j \equiv i \ (\mathrm{mod}\ 2)\}, \quad i = 0, 1.$$

Since $B(a,b)_0$ is a nilpotent algebra it follows that it is a Cartan subalgebra of $B(a,b)$ and $ky_0$ is a maximal toral subalgebra of $B(a,b)$. The maximal subalgebra $P$ of $B(a,b)$ that contains $B(a,b)_0$ has a basis $\{y_i, z_i \mid i \geqslant 0\}$, hence $codim_{B(a,b)}P = 2$. Then by Skryabin's Theorem 6.3 [4] $B(a,b)$ is an algebra of Hamilton type. Since $dim\, B(a,b) = 2^{a+b+1}-2$ it follows that $B(a,b) \simeq H(2,(p,q),\omega_0)^{(2)}$, $p+q = a+b+1$, see [5].

Finally, for any Lie algebra $L$ we can define the invariants (see [2]) $M_n(L) = dim_k\, Span\{(adx)^{2^i} \mid 0 \leqslant i \leqslant n,\ x \in L\}$, where powers $(adx)^{2^j}$ are calculated in $End_k\, L$ and $\xi_n(L) = M_n(L) - M_{n-1}(L)$. In [2] the functions $f_n(x,y)$, $n, x, y \in \mathbf{N}$, were defined, such that

$$f_n(x,y) = 0, \quad \text{if } x, y \leqslant n \quad \text{and} \quad f_n(x,y) = 2, \quad \text{if } x, y > n,$$

in all other cases $f_n(x,y) = 1$.

We have (see [2])

$$\xi_n\big(B(a,b)\big) = 3, \quad \text{if } n = 1; \qquad \xi\big(B(a,b)\big) = f_{n-1}(a-1,b), \quad \text{if } n > 1. \tag{3}$$

But the Hamiltonian algebra $H(2,(a,b+1),\omega_0)^{(2)} \simeq H(2,(b+1,a),\omega_0)^{(2)}$ has the same $\xi_n$-invariants (see [4]), then $B(a,b) \simeq H(2,(a,b+1),\omega_0)^{(2)}$.

Theorem is proved. $\square$

**Corollary 1.** *The algebras $B(a+1,b)$ and $B(b+1,a)$ are isomorphic.*

*The 14-dimensional simple Lie algebras $B(2,1)$ and $K_{14}$ are not isomorphic.*

**Proof.** By Theorem 1 we have $B(a+1,b) \simeq H(2,(a+1,b+1),\omega_0)^{(2)} \simeq H(2,(b+1,a+1),\omega_0)^{(2)} \simeq B(b+1,a)$.

We recall the construction of $K_{14}$ [3]. Let $V$ be a 4-dimensional $\mathbf{F}_2$-space. We can realize $V$ as a set $V = \{\sigma \mid \sigma \subseteq I = (1234)\}$ with the operation $\sigma \Delta \tau = (\sigma \setminus \tau) \cup (\tau \setminus \sigma)$. Then $K_{14} = Span_F\{\sigma \in V\colon\ \sigma \neq \emptyset, \sigma \neq I\}$. Then the multiplication in $K_{14}$ is given by $[\sigma, \tau] = |\sigma \cap \tau|(\sigma \Delta \tau)$. Moreover, it is easy to see that $|\sigma\Delta\tau| - 2 = |\sigma| - 2 + |\tau| - 2$, if $[\sigma,\tau] = \sigma\Delta\tau$. Hence $K_i = F\{\sigma|\colon |\sigma| - 2 = i\}$, $i = -1, 0, 1$ is a $\mathbf{Z}$-grading of $K_{14}$, where $dim\, K_{-1} = dim\, K_1 = 4$, $dim\, K_0 = 6$.

On the other hand every $\mathbf{Z}$-grading of the algebra $H = H(2,(2,2),\omega_0)^{(2)} = \sum_{i\in\mathbf{Z}} \oplus H_i$ has $dim_F\, H_0 \leqslant 4$. Indeed, $H$ has the $\mathbf{Z}^2$-grading, described above, such that $H = \sum_{(i,j)\in\Gamma} \oplus H_{i,j}$, where

$$\Gamma = \bigl\{(i,j)\bigm| -1 \leqslant i, j \leqslant 2;\ (i,j) \neq (-1,-1), (2,2)\bigr\}.$$

Since $H$ does not admit a $\mathbf{Z}^3$-grading, it follows that for any $\mathbf{Z}$-grading of $H$ there exist $a, b \in \mathbf{Q}$ such that $dim_F\, H_0 = |\{(i,j) \in \Gamma\colon\ ai + bj = 0\}|$. Hence $dim_F\, H_0 \leqslant 4$. $\square$

## 2. On the absolute toral rank of simple Lie algebras over a field of characteristic 2

We give a short proof of the following theorem of S. Skryabin [4].

**Theorem 2.** *A finite dimensional simple Lie algebra over a field of characteristic* 2 *has absolute toral rank of at least* 2.

**Proof.** Let $F$ be an algebraically closed field of characteristic 2, let $\tilde{L}$ be a finite dimensional simple Lie algebra over $F$ of toral rank one, let $L$ be a 2-envelope of $\tilde{L}$. Then $L = L_0 \oplus L_1$, where $L_0$ is a Cartan subalgebra of $L$, $[L_0, L_1] \subseteq L_1$ and $[L_1, L_1] \subseteq L_0$. By the definition of absolute toral rank, $L_0$ contains a unique toral element $h = h^{[2]}$. We denote

$$\mathcal{N} = \bigl\{a \in L \bigm| a^{[2^n]} = 0 \text{ for some } n \geqslant 1\bigr\}, \qquad \mathcal{T} = \bigl\{t \in L \bigm| t^{[2]} = t \neq 0\bigr\}.$$

We need the following simple result.

**Lemma 1.** *Let $n \in \mathcal{N}$, $t, s \in \mathcal{T}$.*
*For any $a \in L$, if $a + [a,n] = 0$ then $a = 0$. If $[t,s] = 0$ then $t = s$.*

Note that $L_1 \subseteq \mathcal{N}$. Indeed, if $a \in L_1$, then $a^{[2]} \in L_0$ and if $a^{[2]}$ is not nil then $a^{[2]} = h + n$, where $n$ is a nil element. In this case, $[a, a^{[2]}] = a + [a,n] = 0$, hence $a = 0$ by Lemma 1.

If $H = [L_1, L_1]$ then $\tilde{L} = H \oplus L_1$ and $H$ is not a nil subalgebra. Indeed, if $H$ is a nil subalgebra then by the Jacobson–Engel theorem [1] $\tilde{L}$ is nilpotent since all elements of $L_1$ are nil.

Let $N$ be a maximal nil-ideal of $L_0$ and $N_0 = Fh \oplus N$.

**Step 1.** $L_0 \neq N_0$.

If $L_0 = N_0$, then for some $a, b \in L_1$ we have $[a,b] = h + n$, $n \in N$, since $H$ is not nil. Since $N$ acts nilpotently on $L_1$ we may choose $a$, $b$ with additional property $[[a,N],b], [[b,N],a] \in N$. Hence

$$\bigl[a^{[2]}, b^{[2]}\bigr] = \bigl[\bigl[[a,b],b\bigr],a\bigr] = h + n + \bigl[[b,n],a\bigr] = h + m, \quad m \in N.$$

On the other hand, $a^{[2]}, b^{[2]} \in N$, as we proved above. Hence, $[a^{[2]}, b^{[2]}] \in N$, which contradicts the previous equality. We proved that $L_0 \neq N_0$.

**Step 2.** *$L_0 = L_0' \oplus Ff$, where $L_0' = \{x \in L_0 \mid [x,e] \in N\}$, $e \notin N_0$, $e^{[2]} \in N_0$, $[e, N_0] \subseteq N_0$.*

The factor-algebra $L_0/N_0$ is nil, so it contains a 1-dimensional ideal $F\bar{e}$. Let $e$ be some preimage of $\bar{e}$. It is clear that $e^{[2]} \in N_0$. For some $\gamma \in F$ we have $(e + \gamma h)^{[2]} \in N$. Hence we can suppose that $e^{[2]} \in N$. Denote $L_0' = \{x \in L_0 \mid [x,e] \in N\}$. If $L_0' = L_0$ then $Fe \oplus N$ is an ideal of $L_0$ with nil-radical $Fe \oplus N$, contradicting the maximality of $N$. Hence $L_0 = L_0' \oplus Ff$ for some $f \in L_0$.

**Step 3.** *$F\{X\} = L_0$, where $X = \{a^{[2^i]} \mid a \in L_1,\ i = 1, 2, \ldots\} \subseteq L_0$.*

Since $[a, b] = (a+b)^{[2]} - a^{[2]} - b^{[2]} \in F\{X\}$, when $a, b \in L_1$, we have $H = [L_1, L_1] \subseteq F\{X\}$. Hence $I = F\{X\} \oplus L_1$ is a subspace of $L$ containing the simple ideal $\tilde{L}$. Since the factor algebra $L/\tilde{L}$ is abelian, $I$ has to be an ideal of $L$. In particular, $I$ is a subalgebra. This implies that $I$ is closed under [2]-powers, for it now suffices to check that $x^{[2]} \in I$ when $x \in X$ and when $x \in L_1$. Then $I$ contains the 2-envelope of $\tilde{L}$, that is, $I = L$, which proves Step 3.

**Step 4.** *There exist $a, b \in L_1$ such that $[a, b] = h + n$, $n \in N$.*

By Step 2 $[e, N_0] \subseteq N_0$. Hence, for an arbitrary $x \in L_0$, from $[x, e] \in N = Fh \oplus N_0$ it follows that $[x^{[2]}, e] \in N$, since $N$ is an ideal of $L_0$. Then $L_0'$ is closed under [2]-powers. Hence there exists $c \in L_1$ such that $c^{[2]} \notin L_0'$. Without loss of generality we can suppose that $[c^2, e] = h + m$, where $m \in N$. Now we can choose $a = c$, $b = [c, e]$, in this case we have

$$[a, b] = \alpha^{-1}\big[c, [c, e]\big] = \alpha^{-1}\big[c^{[2]}, e\big] = h + n, \quad n \in N.$$

Step 4 is proved.

**Step 5.** *Let $x = h + a + a^{[2]} + \cdots + a^{[2^k]}$, $y = h + b + b^{[2]} + \cdots + b^{[2^s]}$, $a^{[2^{k+1}]} = b^{[2^{s+1}]} = 0$, $z = [x, y]$. Then $x, y \in \mathcal{T}$, $z^{[2^{r+1}]} = 0$ for some $r$ and $x + y + z + z^{[2]} + \cdots + z^{[2^r]} = 0$.*

A straightforward calculation shows that $x, y \in \mathcal{T}$ and if $z = [x, y]$ then $[x, z] = z$, $[y, z] = z$. We note that $z$ is nil. Indeed, as we proved above for any toral element $t^{[2]} = t$ and corresponding decomposition $L = L_0^t \oplus L_1^t$, $L_i^t = \{x \in L \mid [x, t] = ix\}$, $i = 0, 1$, all elements from $L_1^t$ are nil. But $z \in L_1^x$, hence $z^{[2^{r+1}]} = 0$ for some $r$. Denote $u = x + y + z + z^{[2]} + \cdots + z^{[2^r]}$. It is easy to see that $u^{[2]} = u$. We have $[u, x] = [y, x] + z = 0$, hence $u = x$ or $u = 0$, by Lemma 1. If $u \neq 0$, then $u = x$. Hence $[u, y] = [x, y] + z = z = 0$ and $T = F\{x, y\}$ is a toroidal subalgebra. Then $dim\, T = 1$ and $x = y$. But $x \neq y$ because they have different $L_1$-components $a$ and $b$. Hence $u = 0$.

Step 5 is proved.

**Step 6.** *Let $V = [L_1, N]$ be an $L_0$-submodule of $L_1$. Then $P = V \oplus L_0$ is a proper 2-subalgebra of $L$ and $u \equiv (a + b) \pmod{(P)}$.*

Since $N$ is nil, $V = [L_1, N]$ is a proper $L_0$-submodule of $L_1$. Then $P = V \oplus L_0$ is a proper 2-subalgebra of $L$. We can assume that $a \notin V$. Indeed, if $a \in V$ and $d \notin V, d \in L_1$, then for an arbitrary $\gamma \in F$ we have $f(\gamma) = [(a + \gamma d)^{[2]}, e] = [a^{[2]}, e] + \gamma[[a, d], e] + \gamma^2[d^{[2]}, e]$. If $f(\gamma) \in L_0'$ for all $\gamma \in F^*$ then $[a^{[2]}, e] \in L_0'$, a contradiction. Then $f(\gamma) \notin L_0'$ for some $\gamma \in k^*$ and we can choose $a + \gamma d$ instead of $a$. But $a + \gamma d \notin V$.

Let us prove that $u \equiv (a + b) \pmod{P}$. We have by definition,

$$z = a + b + \big[a, b^{[2]}\big] + \big[b, a^{[2]}\big] + \sum_{i>1}\big[a, b^{[2^i]}\big] + \big[b, a^{[2^i]}\big] + z_0, \tag{4}$$

where $z_0 \in L_0$. But

$$[a, b] = h + n, \qquad \big[a, b^{[2]}\big] = [h + n, b] = b + [b, n], \qquad \big[b, a^{[2]}\big] = a + [a, n],$$
$$\big[a, b^{[4]}\big] = \big[[b, n], b^{[2]}\big] = \big[b, \big[n, b^{[2]}\big]\big] \in V.$$

By induction, for $i > 1$, $[a, b^{[2^i]}] = [[n, b^{[2^{i-1}]}], b^{[2^{i-1}]}] \in V$. Hence, by (4), $z \in P$. Then $u \equiv x + y \pmod P \equiv a + b \pmod P$.

Step 6 is proved.

Now we can finish the proof of Theorem 2. We have $u = a + b + z + \cdots \equiv (a + b) \pmod P$. By Step 5, $u = 0$. It means that $a \equiv b \pmod P$. By choice of $a$, $b$ we have $b = [a, e]$ and $[b, e] = [a, e^{[2]}] \in V$, since $e^{[2]} \in N$. Since $[a, e] = b \equiv [b, e] \equiv 0 \pmod P$ it follows that $a \equiv 0 \pmod P$, a contradiction. $\square$

Note that S. Skryabin [4] obtained a much deeper result. He described all simple finite dimensional Lie algebras over a field $F$ that contain some Cartan subalgebra of toral rank 1.

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